Reflections on Modeling From an Unlikely Modeler

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Prologue…

Let me begin at the beginning…

and hint at the end.

1940, G.H. Hardy:

I have never done anything ‘useful’. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.

1975, A. H. Schoenfeld:

PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

CONTINUOUS MEASURE-PRESERVING MAPS ONTO PEANO SPACES

ALAN H. SCHOENFELD

In this paper we obtain measure-theoretic versions of some topological existence theorems relating to continuous maps, in particular the Hahn-Mazurkiewitz Theorem. Let X be a Peano Space and λ a Borel measure on X with $\lambda(X)=1$. There is a continuous measure-preserving surjection from the unit interval (with Lebesque measure) to X if and only if the support of λ is X.

1. Introduction. Peano's construction in 1890 of a continuous surjection from $I=[0,1]$ to $S=I\times I$ led twenty-five years later to the Hahn-Mazurkiewitz theorem, which characterizes those topological spaces which are the continuous image of the unit interval.[1] Since the original paper, there have been many elegant constructions of space-filling curves, including one by Hilbert.[2] Perhaps most striking about Hilbert's construction is its remarkable symmetry: one sees easily that for each of the intervals $A_{i,j}=[i/4^j, (i+1)/4^j]$, the image of $A_{i,j}$ is a square of area 4^{-j}; and the images of distinct intervals $A_{i,j}$ and $A_{k,j}$ intersect in a set of (planar Lebesgue) measure zero. It is easy, in fact, to verify that Hilbert's space-filling curve is measure-preserving. This suggests the possibility that, under suitable restrictions, a Peano space X which is also a measure space might be the image of a continuous measure-preserving map from the unit interval. Clearly a necessary condition for the existence of such a map is that open subsets of X have positive measure. The aim of this paper is to show that this condition is sufficient as well. We will prove

THEOREM 1. *Let $I=[0,1]$ and μ be Lebesgue measure on I. Let X be a Peano space and λ a Borel measure on X with $\lambda(X)=1$. Then a* (necessary and) *sufficient condition that there be a continuous measure-preserving surjection $f:\langle I,\mu\rangle\to\langle X,\lambda\rangle$ is that X be the support of λ*[3].

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1. The Hahn-Mazurkiewitz theorem states:

A topological space X is the continuous image of $I=[0,1]$ if and only if X is compact, connected, locally connected, and metrizable. Such spaces are called Peano spaces.

2010, Deborah Ball to Alan Schoenfeld:

I don't understand your pathological need to model everything.

Why would you do that?

2010, Alan Schoenfeld to Deborah Ball:

Theorems tell you how things work, and why. When I left mathematics I left more than absolute truth - I lost a way of nailing down how things work.

It’s easy to bullshit people when you make claims about people and things.

I’m great at bullshit.

So, I need a replacement for the mathematics to keep me honest.

2010, Alan to Deborah, Part 2:

Models are the way I test my understanding.

If I put certain values in for the parameters, does the model behave the way it “should”? If it doesn’t, then I’m missing something (maybe a parameter, maybe a relationship).

Models are falsifiable.

You can’t just wave your hands. With a specific model, you can make predictions, and how good they are tells you how good your understanding is.

Part 1
Why I talk about sense making.

I begin with a series of examples, which have various connections to modeling and the “real world”

Not Sense-Making, 1: NAEP, 1985

An army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed?

The Most Common Answer:

31R12

Not Sense-Making, 2:

How many two-foot boards can be cut from two five-foot boards?

Not Sense-Making, 3:

Kurt Reusser asks 97 1st and 2nd graders:

There are 26 sheep and 10 goats on a ship.
How old is the captain?

76 students "solve" it, using the numbers.

Not Sense-Making, 4:

H. Radatz gives non-problems such as:

"Mr. Lorenz and 3 colleagues started at Bielefeld at 9 AM and drove the 360 km to Frankfurt, with a rest stop of 30 minutes."

At every grade level, more students combined numbers and gave an answer!

The point, from my problem solving work:

These are examples of student **beliefs**, developed from students' experience in the mathematics classroom.

The way in which students engage with the mathematics is every bit as important – maybe more – than the content itself.

To hammer the point home:

Have you seen what they did to "problem solving" in the 1980s?

"If you see a problem with an ***n*** in it, substitute values for ***n*** = 1,2,3,4 and find the pattern."

Here are some problems to try.

1. What is the sum of the first n odd numbers?
2. …

That is NOT problem solving!

It’s teaching rote mathematics in the name of problem solving.

What follows are some slides showing what is being done today in Pólya’s name.


Problem
Read your problem.
Analyze
Underline key words. Cross out information you do not need. Decide which operation you will need to use.
Work it out
Use the operation to work the problem out. Show your work.
Solve
Write your answer. Ask yourself, "Does my answer make sense?"

RIDE – Math Problem Solving Format

Read: What is this problem asking you to do? (write the question)

____

____

Identify: What information do you have to solve this problem? (write the important information)

____

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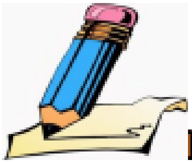

Determine the operation: Describe the operation

____

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____

Enter the Numbers and Calculate: Set up the problem, solve and check!

We call the process **PIES**. The acronym stands for

Picture - draw a simple picture showing the situation

Information - find important facts (information) and list them next to the picture

Equation - find an equation which match the information on the picture

Solve - insert information from picture into equation and solve algebraically

F.U.S.E.

F Find the important numbers and words.

U Understand what the problem is asking.

S Select a strategy and tools to solve the problem.

E Explain your thinking and prove your answer.

Elementary AMC

Read and Record the problem.

Illustrate your thinking.

Compute.

Explain your thinking.

Someone asked me what to do about this.

I said,
"That's totally stupid. Morons want algorithms when the idea is to think. Here are some resources to do it right."

But also, here's a dumb rubric, if it would make them happy. Please send me the royalties":

***T**ake the time to understand what the problem is all about.*

***H**ave the big picture in mind, attending to all relevant variables.*

***I**nvestigate ways to represent the situation symbolically, so you can solve it.*

***N**ote whether your solution methods match the situation described*

***K**eep a close eye on your solution, checking it carefully.*

I shared with some friends.

Hugh Burkhardt tried to refine the language.

I said that wasn’t the point, and created a really sarcastic version.

Bill McCallum got it, and suggested a “preparation for problem solving” preface.

Here’s what collaboration can do:

Problem Solving Made Simple

Prior to engaging,

Brainstorm methods of solution

Understand all the concepts involved

Link the concepts together

Learn the techniques that will be used

Once you're ready,

Start by thinking the situation through

Have meaningful representations available

Interpret symbolic manipulations

Take care to review your work

Think it can’t happen here?

Modeling is a closely related to the scientific method.

Here is what I got when I used google images on the scientific method.

The Scientific Method as an Ongoing Process
The Scientific Method
observation
question
Hypothesis
Research
Data
Experiment
Conclusion
Materials
Procedure
Hypothesis
Conclusion
Statement
SCIENTIFIC METHOD
PURPOSE
RESEARCH
HYPOTHESIS
EXPERIMENT
ANALYSIS
CONCLUSION
Scientific Method
HYPOTHESIS
EXPERIMENTATION
REFINE THE IDEA
EXPERIMENTATION
FINAL STATEMENT
The Scientific Method
Question
Hypothesis
Procedure
Experiment
Data Analysis
Conclusion
Abstract
Presentation
scientific method
research
hypothesis
THE SCIENTIFIC METHOD
Question:
Why won't my toaster toast?
Observation:
The toaster won't toast!
Hypothesis:
Maybe the outlet is broken.
Make Observations
Research the Topic to Find Out What Is Already Known
Ask a Testable Question
Turn the Question into a Hypothesis
Make Predictions
Conduct an Experiment to Test the Predictions
Analyze the Results to Arrive at a Conclusion
The Scientific Method
THE SCIENTIFIC METHOD
ASK A QUESTION OR ADDRESS A PROBLEM
RESEARCH
HYPOTHESIS
EXPERIMENT
ANALYSIS
CONCLUSION
Observation
Hypothesis
Testing
Communicate
Question
Prediction
Conclusion
Prediction:
If I plug the toaster into a different outlet, then it will toast the bread.
THEORY
observations
hypothesis
predictions
Scientific Method
STEP 1: Ask a question
STEP 2: State a hypothesis
STEP 3: Conduct an experiment
STEP 4: Analyze the results
STEP 5: Make a conclusion
SCIENTIFIC METHOD
The Scientific Method
Scientific Method
Defined: Series of steps to collect information or solve problems
1) Problem/Observation
2) Hypothesis
3) Experimentation
4) Data Analysis
5) Conclusion
6) Repeat
Think Like A Scientist
Analyze Data
Draw Conclusion
State Problem
Scientific Method
Collect Data
Formulate Hypothesis
Design Experiment
730 × 945 - foodsciencese...
THE SCIENTIFIC METHOD
PURPOSE
HYPOTHESIS
PROCEDURE
OBSERVATION
CONCLUSION
REPORT RESULTS
SCIENTIFIC METHOD
The Scientific Method
Observation / Question
Research
Hypothesis
Experiment
Collect Data
Analysis
Conclusion
Scientific Method
Engineering Method
Question
Hypothesis
Experiment
Observation
Analysis
Conclusion

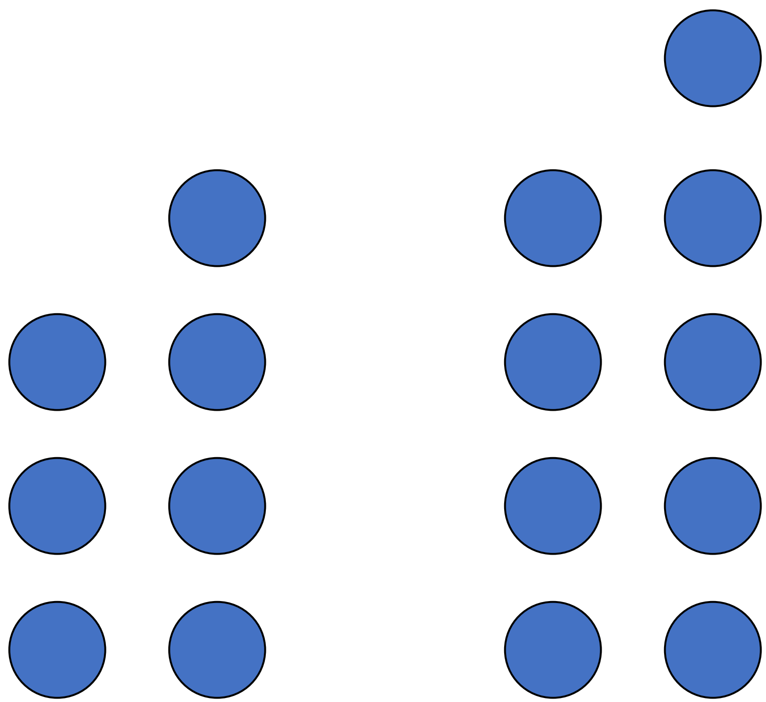
Be nervous what you ask for.

Be very nervous.

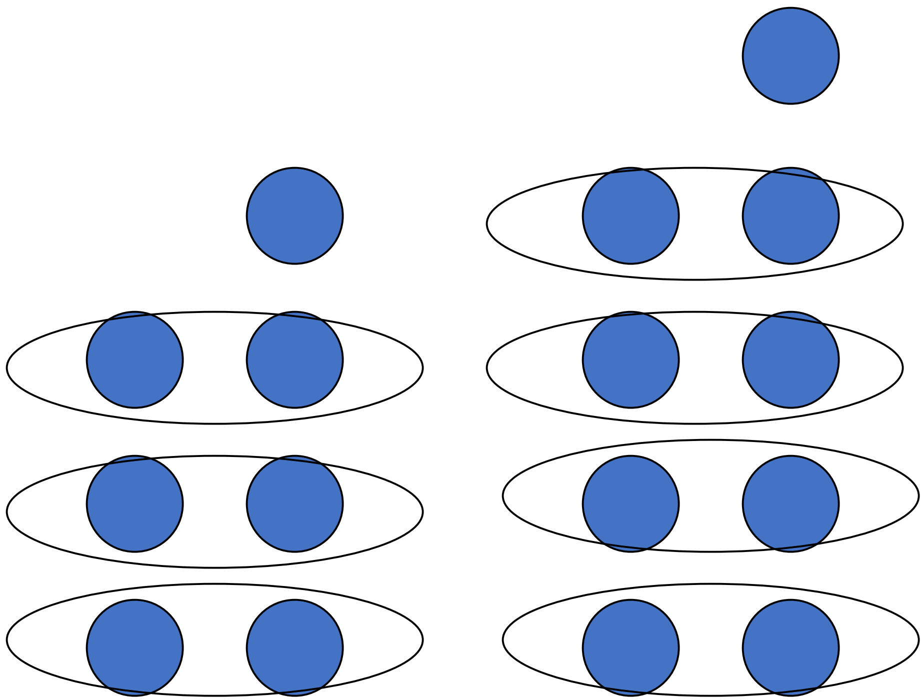
Now, a quick discussion of Sense-Making

What happens when you add two odd numbers?

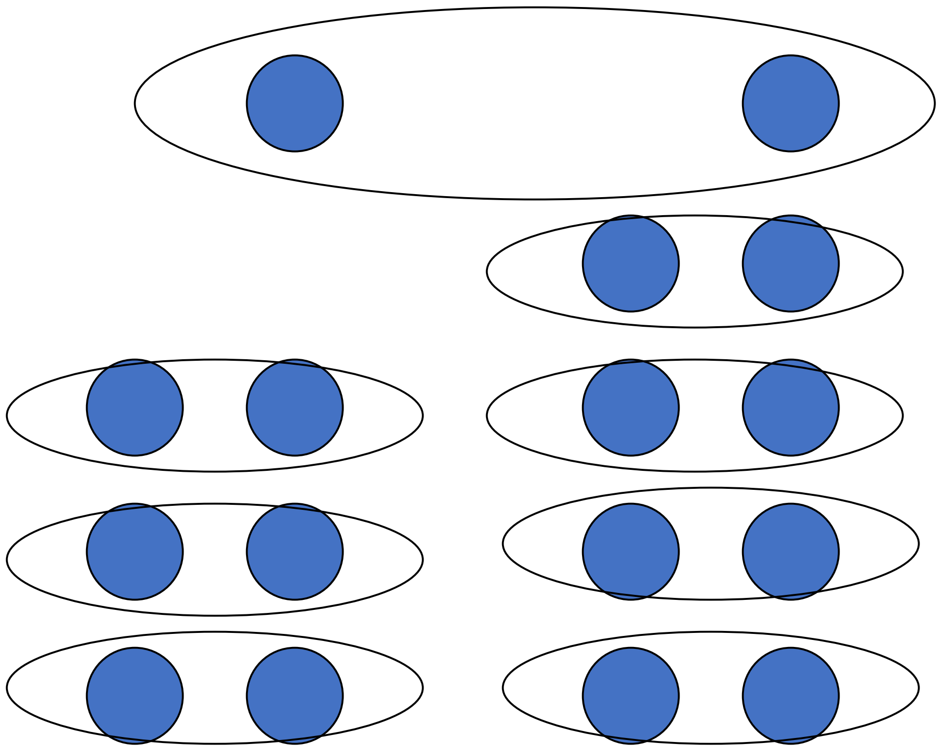
7 + 9



7 + 9



7 + 9



What is this an example of?

1. Elementary arithmetic.
2. Mathematical sense making.

3. MODELING, DAMMIT!

Part 2

Alan as a friend of modeling, as I understand it to be proposed and supported by this community.

The Main Evidence:

The blueprint for the smarter balanced assessments, which I wrote with Hugh Burkhardt

The four Dimensions of SBAC Assessments

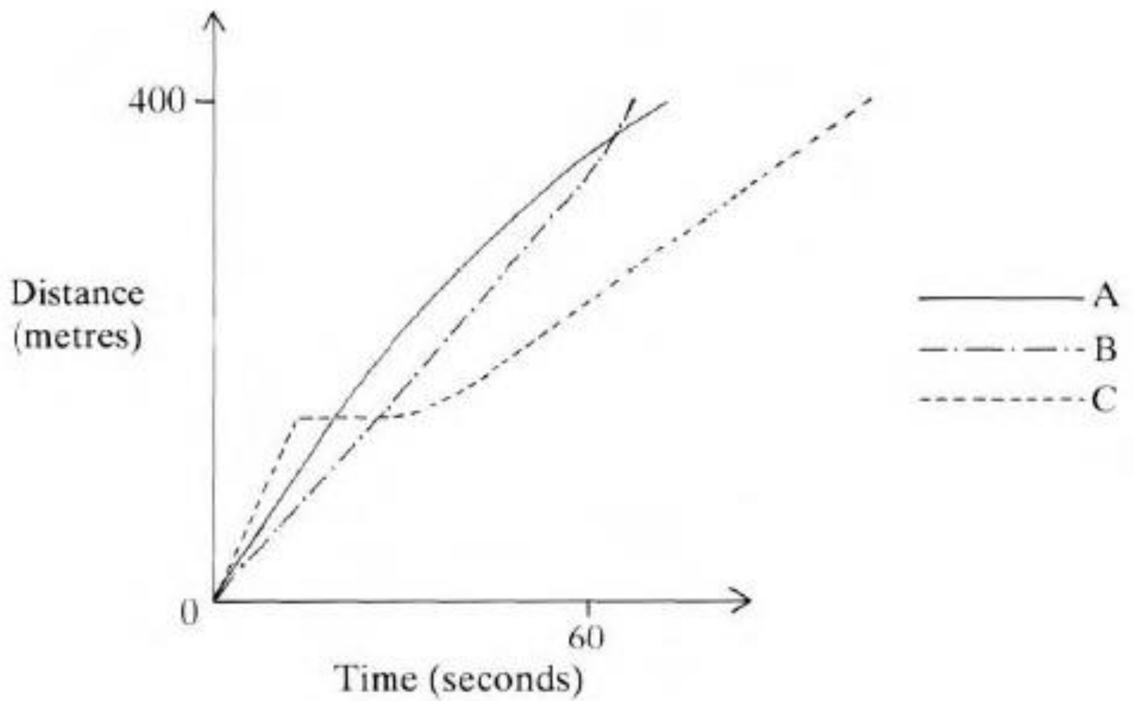
1. Students can explain and apply mathematical concepts and interpret and carry out mathematical procedures with precision and fluency.
2. Students can solve a range of complex well-posed problems in pure and applied mathematics, making productive use of knowledge and problem solving strategies.
3. Students can clearly and precisely construct viable arguments to support their own reasoning and to critique the reasoning of others.
4. Students can analyze complex, real-world scenarios and can construct and use mathematical models to interpret and solve problems.

Total Score for Mathematics			
Content and Procedures Score	Problem Solving Score	Communicating Reasoning Score	Mathematical Modeling Score
40%	20%	20%	20%

Our intention was to force curricular change by having large parts of the exam we didn’t typically test.

Here are some examples

"Hurdles Race."



The rough sketch graph shown above describes what happens when 3 athletes A, B and C enter a 400 metres hurdles race.

Imagine that you are the race commentator. Describe what is happening as carefully as you can. You do not need to measure anything accurately.

Think of the Content involved:

- Interpreting distance-time graphs in a real-world context
- Realizing "to the left" is faster
- Understanding points of intersection in that context (they're tied at the moment)
- Interpreting the horizontal line segment
- Putting all this together in an explanation

Think of the Practices involved:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments…
- Model with mathematics…
- Not a classic “modeling” problem, but well worth doing!

25% Sale, Part 1

In a sale, all the prices are reduced by 25%.

Julie sees a jacket that cost $32 before the sale.

How much does it cost in the sale?

25% Sale, Part 2

In the second week of the sale, the prices are reduced by 25% of the previous week's price.

In the third week of the sale, the prices are again reduced by 25% of the previous week's price.

In the fourth week of the sale, the prices are again reduced by 25% of the previous week's price.

Alan says that after 4 weeks of these 25% discounts, everything will be free. Is he right? Explain your answer.

Again:
Core content, central practices.
Maybe not classical modeling,
but…

And, we had more classical modeling tasks too.

Baseball Jerseys

PRINT IT

Get your baseball jerseys printed with your own team names here.

Only $21 per jersey.

TOP PRINT

We will print your baseball jerseys - just supply us with your design.

Pay a one-off setting up cost of $45; we will then print each jersey for only $18!

VALUE PRINTING

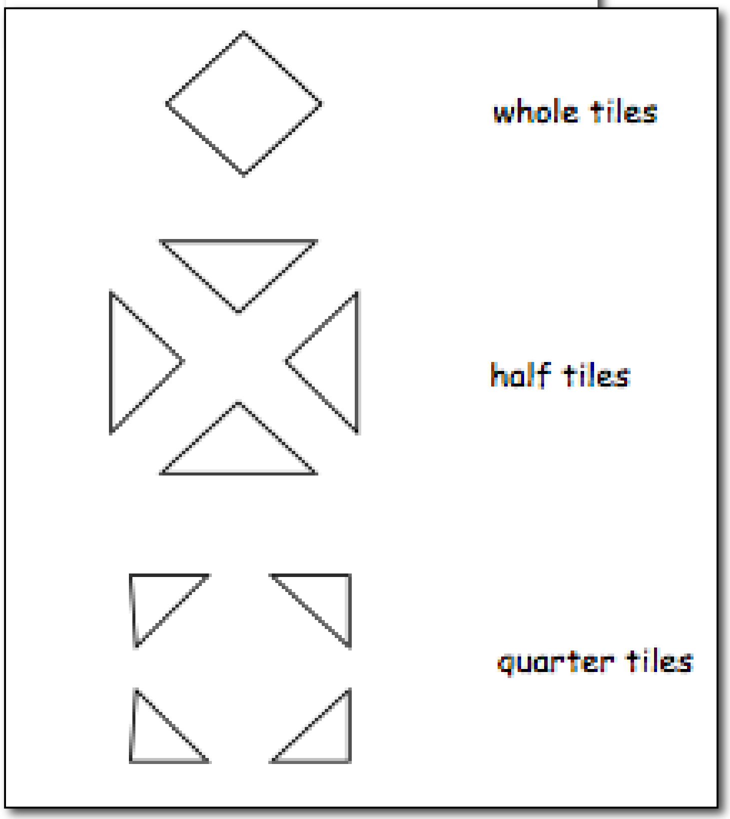
We print baseball jerseys.

Pay a one-off set up cost of $...............

Then each jersey will cost $.................

1. Give Bill some advice. When should he choose PRINT IT? When should he choose TOP PRINT?

2. VALUE PRINTING never wants to be the most expensive and never wants to be the cheapest. How can it arrange that?



Maria makes square tables, then sticks tiles to the top.

Square tables have sides that are multiples of 10 cm.

Maria uses quarter tiles at the corners and half tiles along edges.

How many tiles of each type are needed for a 40 cm x 40 cm square?

Describe a method for quickly calculating how many tiles of each type are needed for larger, square table tops.

Want to see more?

Check out the SBAC specs;

look at

The *Mathematics Assessment Project* (google the name or go to http://map.mathshell.org/materials/)

Part 3
(now that you see I'm your friend):
A distal view of modeling.

I would posit that the "modeling cycle" and instruction on it bear the same relationship to real modeling as the four-step processes I showed earlier bear to real mathematical thinking and sense making.

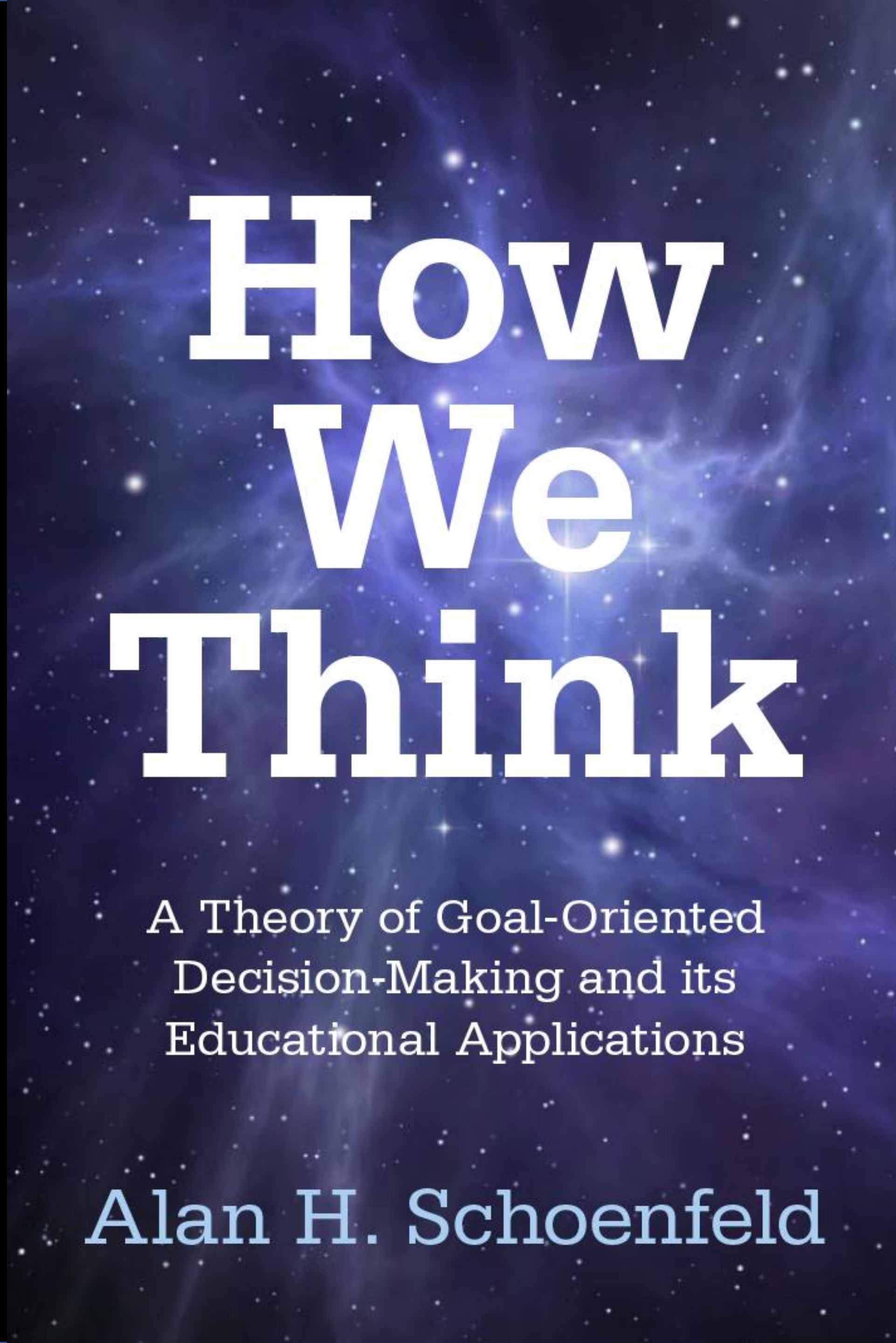
I mentioned earlier that I model like mad.

I spent nearly 20 years developing a theoretical model of people's decision making – especially during teaching – that I used to model and predict individual teachers' classroom decisions.

I will not take you through the detail.

(Say thanks!)

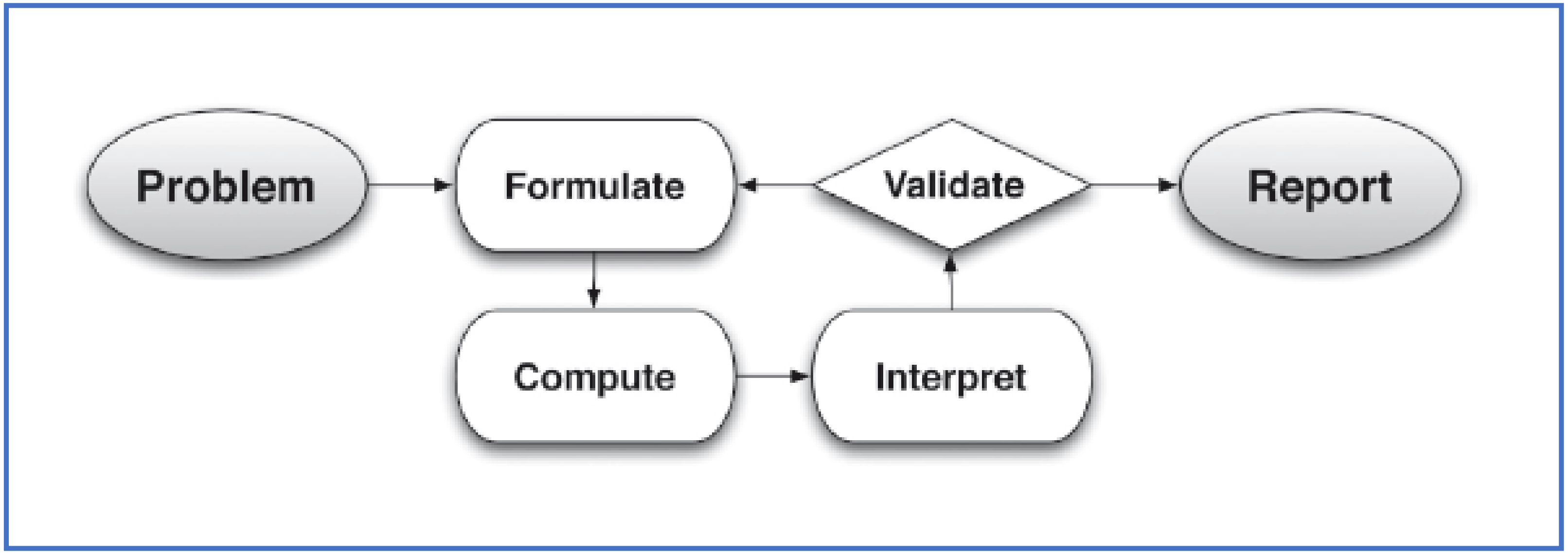
But simply refer you to my book…


How
We
Think
A Theory of Goal-Oriented
Decision-Making and its
Educational Applications
Alan H. Schoenfeld

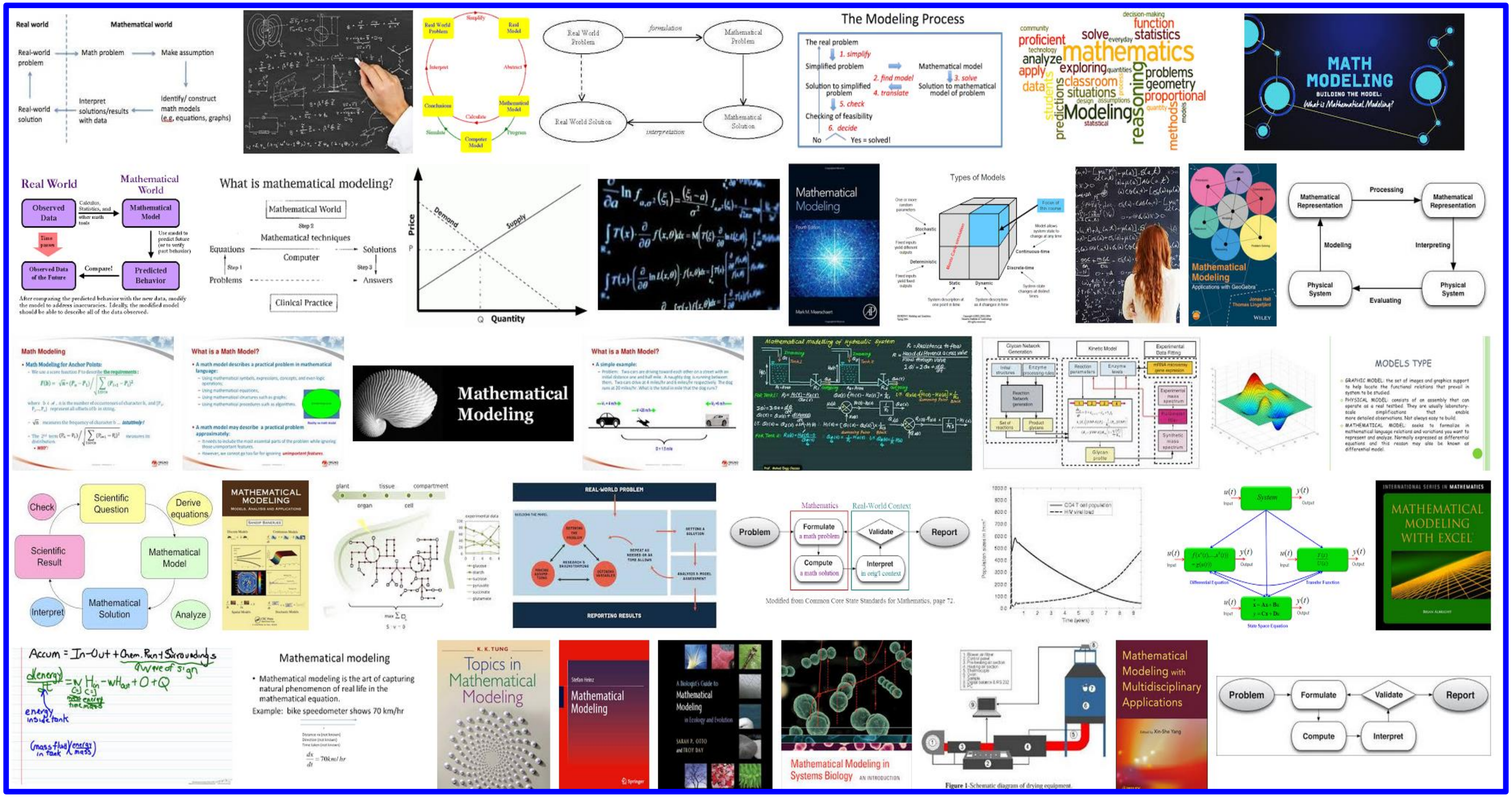
What I want to do instead is take you through my understanding of the modeling process.

There are various representations of the process:

From the US Common Core Standards:

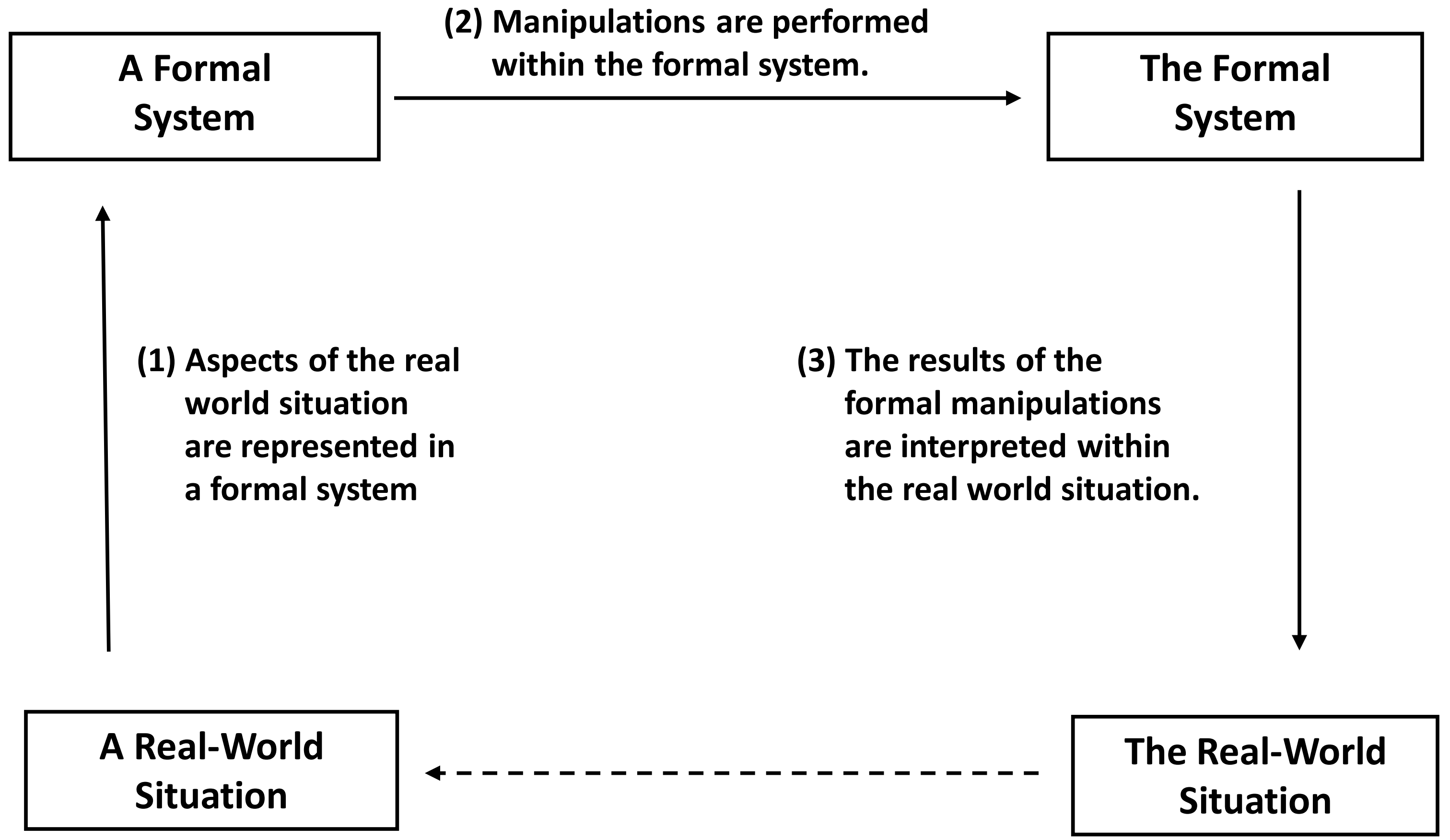


From Google images:



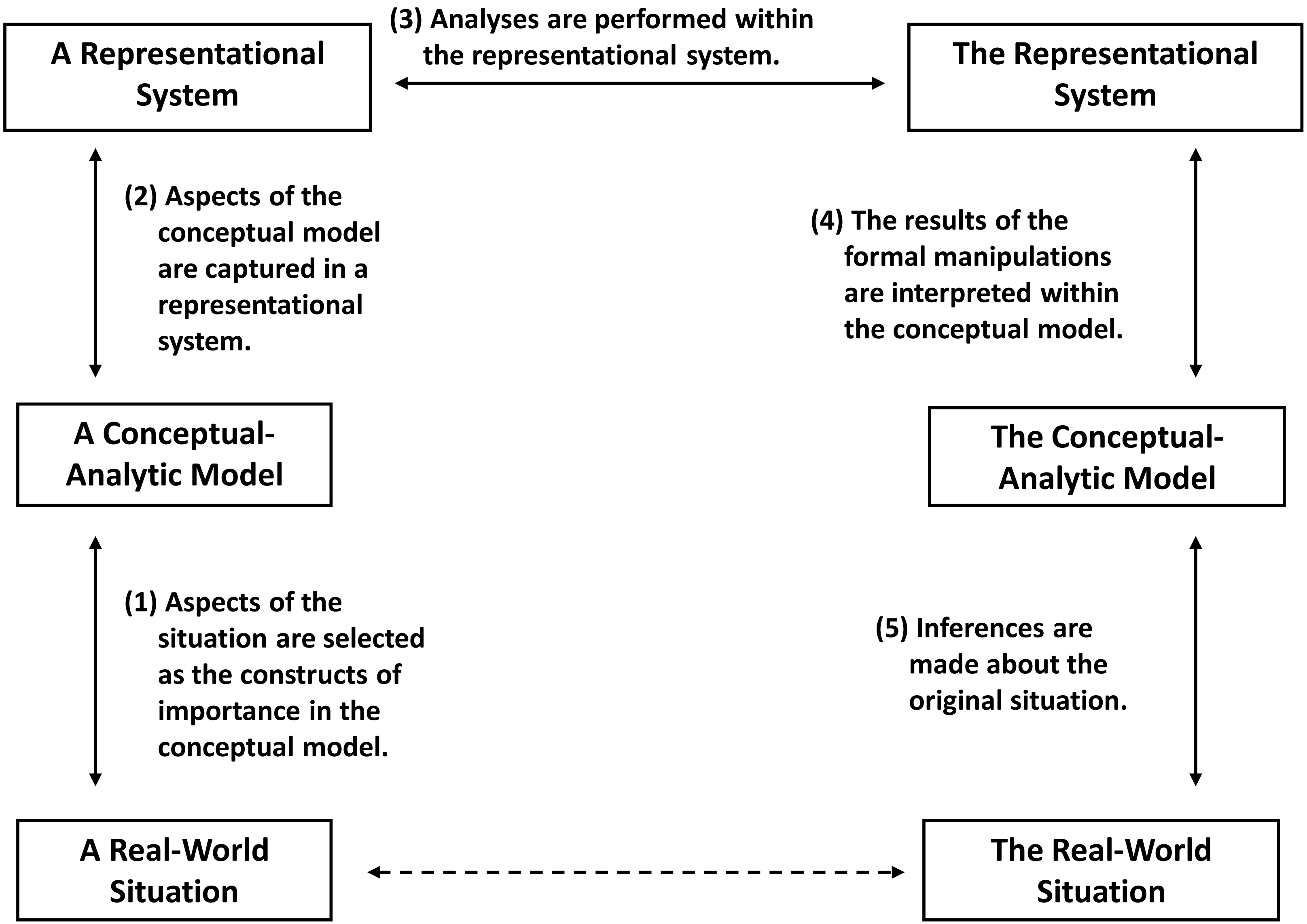
I’m going to be simple-minded.

The next slide shows how most people (including mathematicians and scientists) tend think of the representation and modeling process.


A Formal System
(2) Manipulations are performed within the formal system.
The Formal System
(1) Aspects of the real world situation are represented in a formal system
(3) The results of the formal manipulations are interpreted within the real world situation.
A Real-World Situation
The Real-World Situation

The reality is much more complex.

This is the process as a whole:


A Representational System
(3) Analyses are performed within the representational system.
The Representational System
(2) Aspects of the conceptual model are captured in a representational system.
(4) The results of the formal manipulations are interpreted within the conceptual model.
A Conceptual-Analytic Model
The Conceptual-Analytic Model
(1) Aspects of the situation are selected as the constructs of importance in the conceptual model.
(5) Inferences are made about the original situation.
A Real-World Situation
The Real-World Situation

There are challenges and complexities at every step.

I will discuss the transitions one at a time.

The First Transition

A Conceptual- Analytic Model

(1) Aspects of the situation are selected as the constructs of importance in the conceptual model.

A Real-World Situation

What we *select* in order to frame and conceptualize the "real world" situation in theoretical terms is fundamentally important.

For example:

- Who do the "subjects" represent?

  (Think of medical studies, of men only)

- What "counts" as understanding?
- What classroom processes are of interest?

The Second Transition

A Representational System

(2) Aspects of the conceptual model are captured in a representational system.

A Conceptual- Analytic Model

The nature of the representation – what it captures, and how it works – is of central importance.

For example:

Will you use graphs, tables, flow charts, or…?

How does the representation limit you, both in terms of what it “captures” and what you can “see” in it?

How will you work with it?

Consider these two problems.

<u>Problem 1:</u>
Compute the mean, median, and standard deviation of these two distributions, and represent them graphically:

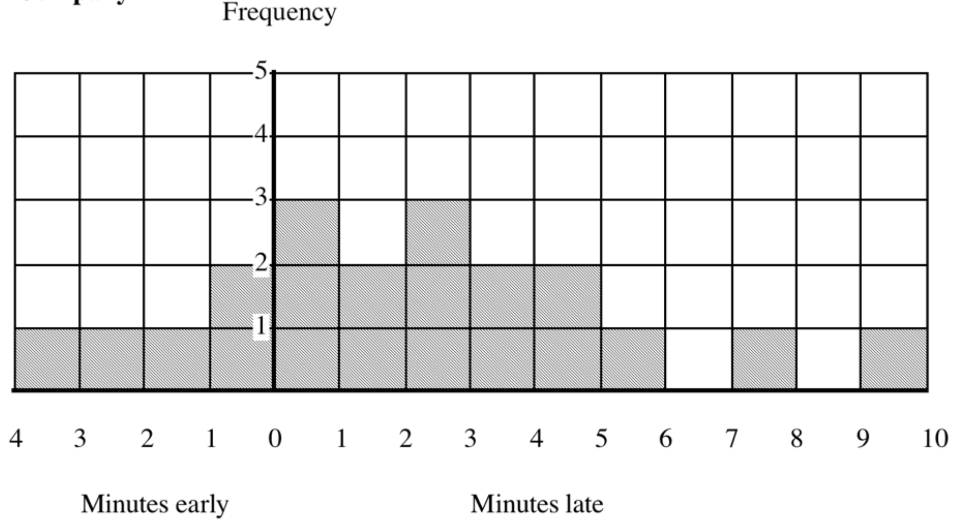
a. -3.5, .75, 1.5, 4.5, -.75, -2.5, 4.75, 2.75, .5, -1.5, 2.25, 9.25, 3.5, 1.25, -.5, 2.5, .5, 7.25, 5.5, 3;
b. 3.75, 4.5, 3, 5, 2.25, 1.25, .75, 3, -.5, 1.5, 3.5, 6, 4.5, 5.5, 2.5, 4.25, 2.75, 3.75, 4.75

Data:

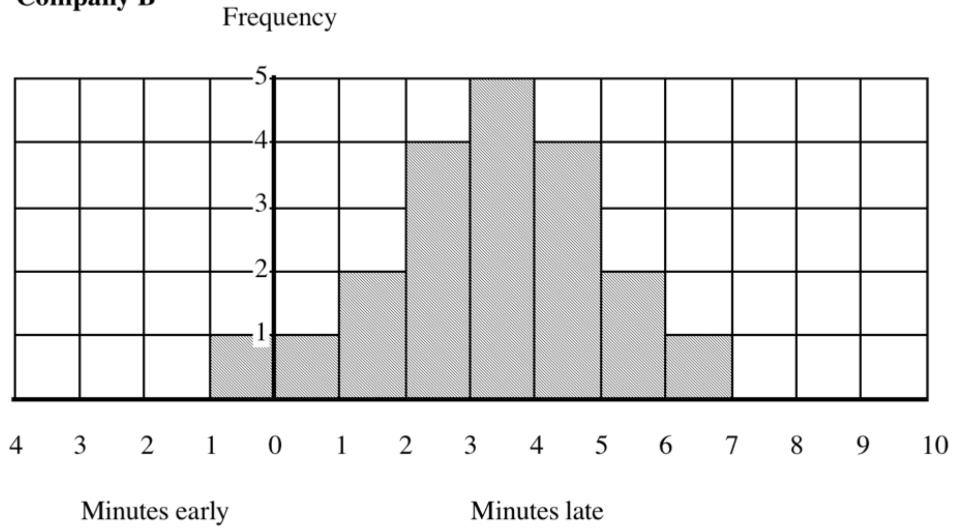
	Company A	Company B
Mean	2 mins 3 secs	3 mins 14 secs
Median	1 min 53 secs	3 mins 15 secs
Range	12 mins 45 secs	6 mins 30 secs
SD	3 min 11 secs	1 min 40 secs

Graphs:

Company A



Company B



This is a complete answer.

Problem 2:

You work for a business that has been using two taxicab companies, Company A and Company B.

Your boss gives you a list of (early and late) "arrival times" for taxicabs from both companies over the past month.

Your job is to analyze those data using charts, diagrams, graphs, or whatever seems best. You are to:

i. make the best argument that you can in favor of Company A;

ii. make the best argument that you can in favor of Company B;

iii. write a memorandum to your boss that makes a reasoned case for choosing one company or the other, using the relevant mathematical tools at your disposal.

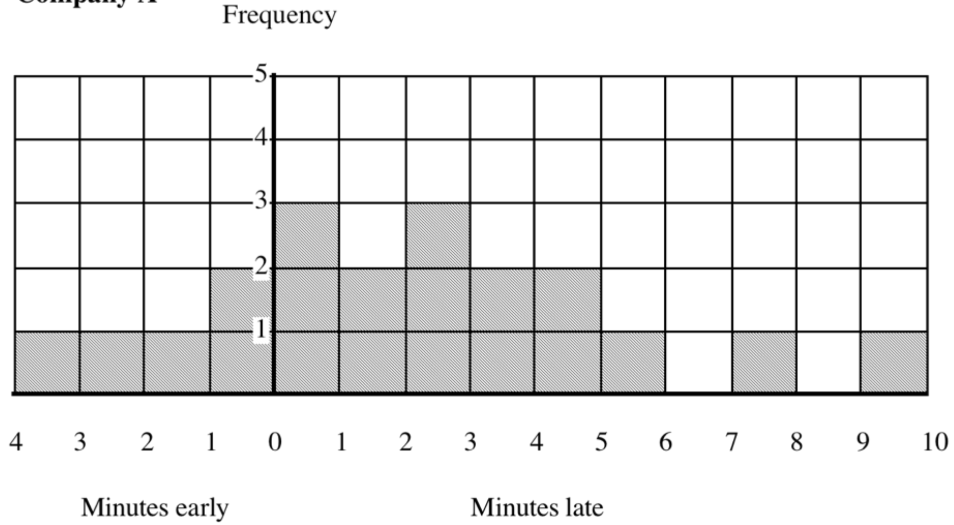
Company A		Company B	
3 mins 30 secs	Early	3 mins 45 secs	Late
45 secs	Late	4 mins 30 secs	Late
1 min 30 secs	Late	3 mins	Late
4 mins 30 secs	Late	5 mins	Late
45 secs	Early	2 mins 15 secs	Late
2 mins 30 secs	Early	2 mins 30 secs	Late
4 mins 45 secs	Late	1 min 15 secs	Late
2 mins 45 secs	Late	45 secs	Late
30 secs	Late	3 mins	Late
1 minute 30 secs	Early	30 secs	Early
2 mins 15 secs	Late	1 min 30 secs	Late
9 mins 15 secs	Late	3 mins 30 secs	Late
3 mins 30 secs	Late	6 mins	Late
1 min 15 secs	Late	4 mins 30 secs	Late
30 secs	Early	5 mins 30 secs	Late
2 mins 30 secs	Late	2 mins 30 secs	Late
30 secs	Late	4 mins 15 secs	Late
7 mins 15 secs	Late	2 mins 45 secs	Late
5 mins 30 secs	Late	3 mins 45 secs	Late
3 mins	Late	4 mins 45 secs	Late

Same Data:

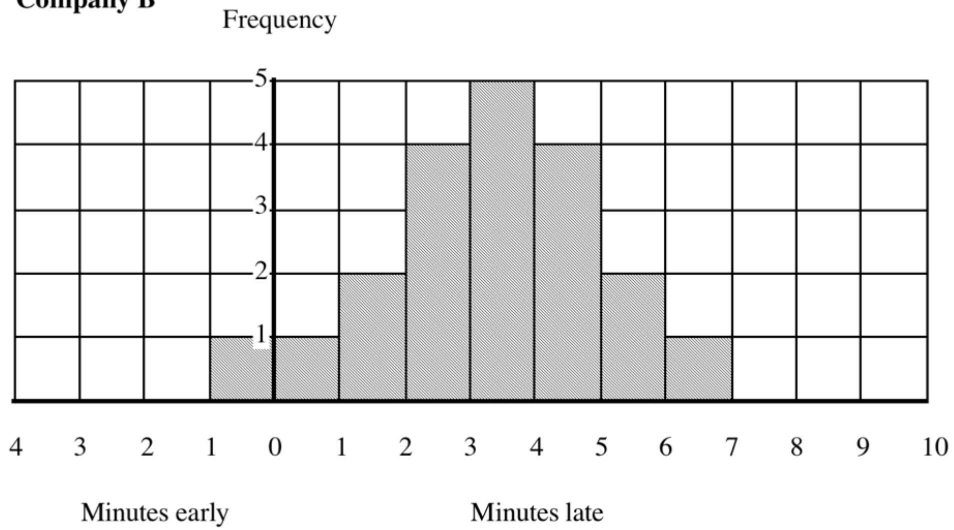
	Company A	Company B
Mean	2 mins 3 secs	3 mins 14 secs
Median	1 min 53 secs	3 mins 15 secs
Range	12 mins 45 secs	6 mins 30 secs
SD	3 min 11 secs	1 min 40 secs

Same graphs:

Company A



Company B



But here’s the right kind of answer:

Company A's cabs are earlier on average than Company B's, but they are less consistent in their arrival times. It's better to order a cab from Company B - but order it for 5 minutes early, so it arrives when you need it.

Both questions test statistical competency, but in *very* different ways. Students can do well on the first (skills) and not on the second (understanding and use).

The Third Transition

A Representational System

The Representational System

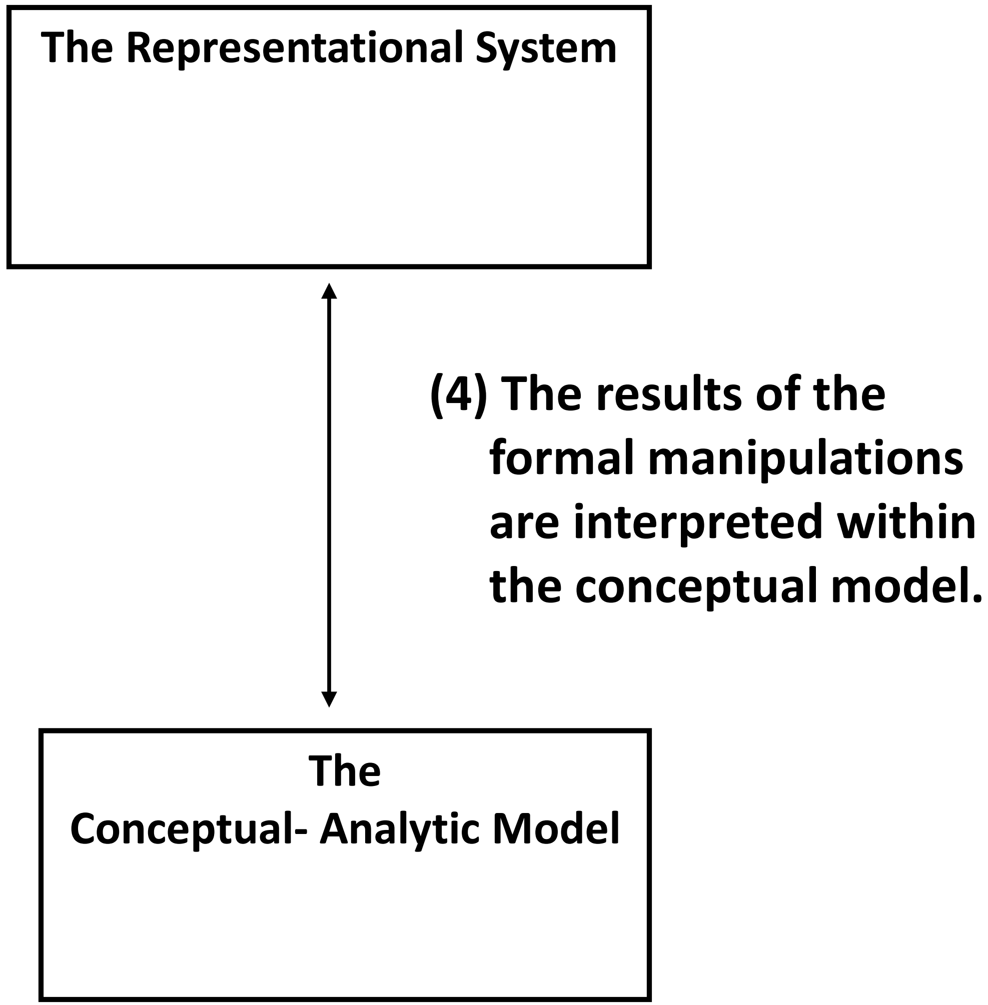
(3) Analyses are performed within the representational system.

The nature of the analyses (and their suitability and interpretation) may or may not be appropriate.

For example:

- Do the conditions for analysis match the assumptions in the model?
- Are the phenomena captured in robust and meaningful ways?
- Are the analytic methods stable (e.g., do they have good inter-rater reliability)?

The Fourth Transition



“Mapping back” to the conceptual model may or may not make sense.

For example:

- Sampling error - Dewey beats Truman!
- Construct validity - e.g., what is “IQ” or “Power relationships, or “self concept?
- Econometric analyses - do proxies for “the ratio of instructional staff to students” really make sense?

The Fifth Transition

The Conceptual- Analytic Model

(5) Inferences are made about the original situation.

The Real-World Situation

The application of the process to the "real world situation" can be problematic.

For example:

- Do psychometrically defined entities such as "verbal or spatial ability" make sense?
- Can ideas such as "sociomathematical norms, or "knowledge," "goals," "beliefs," be defined and used in meaningful ways?
- What's the unit of analysis? A reporter at *Science* misunderstands completely.

A superintendent did a study where he compared three “experimental” schools with three “control” schools, and got no differences on average.

It turns out that one of the schools worked with the program and got great results, one was so-so, and one resisted the program and got lousy results.

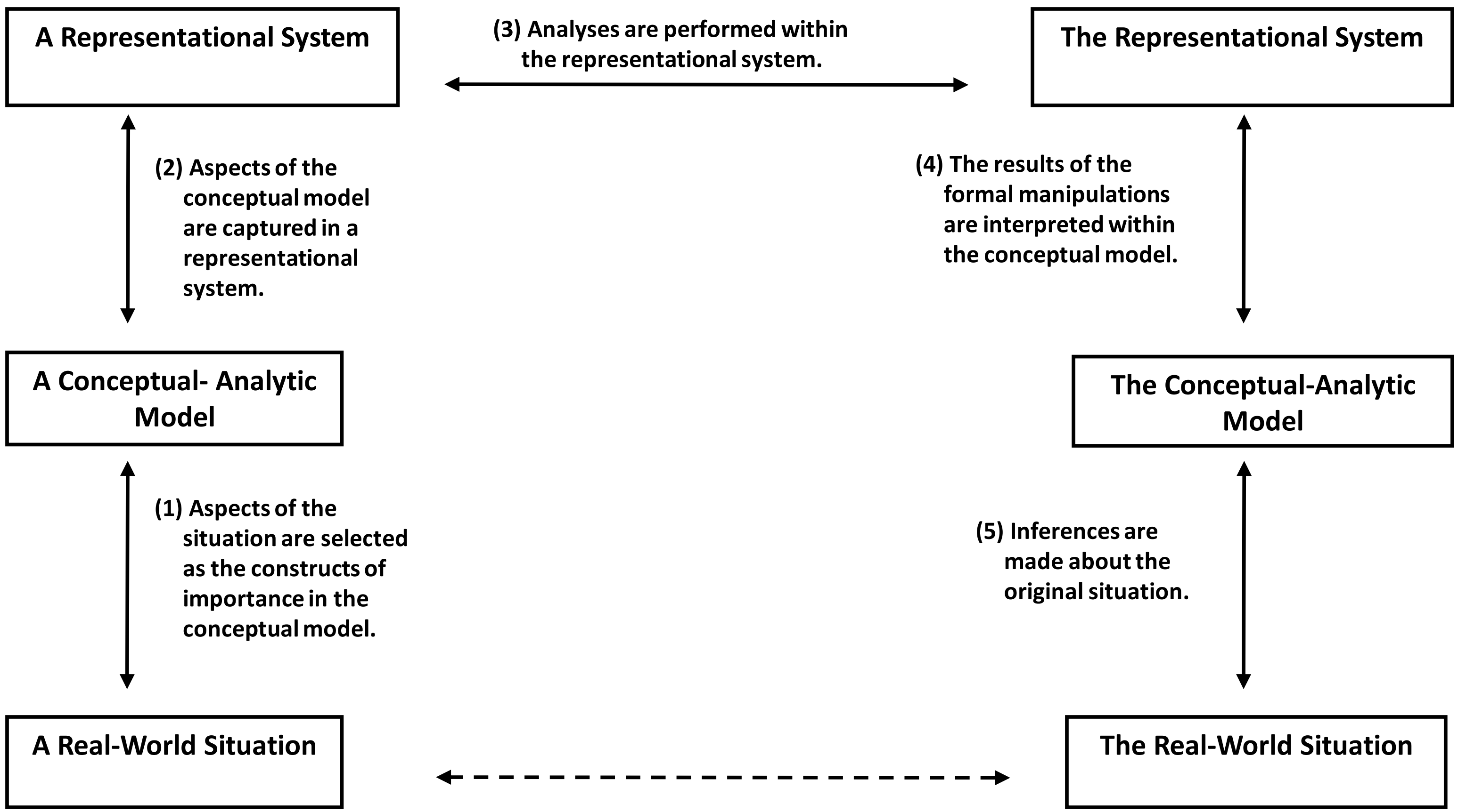
What’s the moral of the story?

The reporter said, “there was no difference between treatments.”

But as I see it:

There are three conditions: supportive, neutral, and negative. If the conditions are supportive, use the new curriculum. If they’re neutral, provide training and encouragement. If they’re negative, stay with the old treatment.

In sum, every aspect of this diagram:

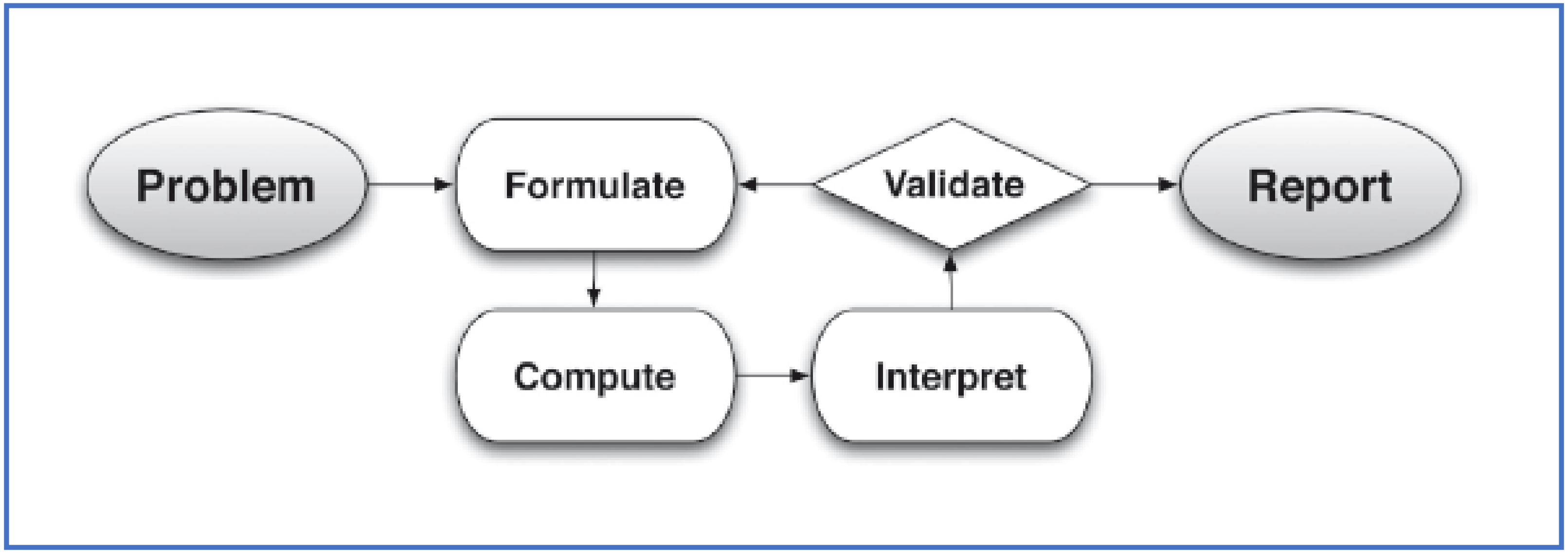


must be considered with great care. That's true in our own research, but equally true when students learn to model.

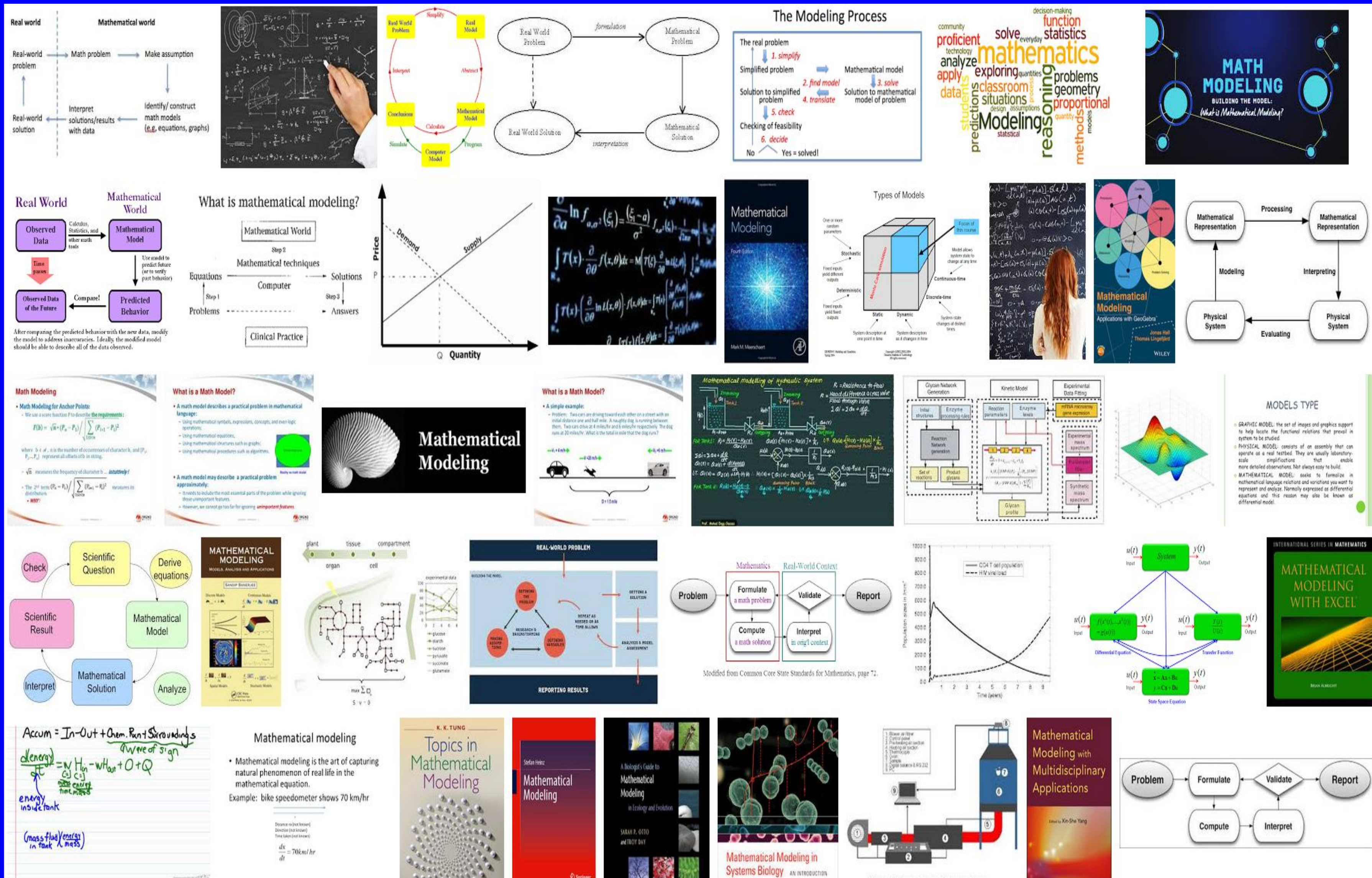
The point is that *this* is the kind of thinking we need to teach students to engage in,

not

This



Or we’ll get this:



Real world
Mathematical world
Real-world problem
Math problem
Make assumption
Real-world solution
Interpret solutions/results with data
Identify/ construct math models (e.g. equations, graphs)
Real World Problem
Mathematical Problem
formulation
Real World Solution
Mathematical Solution
interpretation
The Modeling Process
The real problem
1. simplify
Simplified problem
Mathematical model
2. find model
3. solve
Solution to simplified problem
Solution to mathematical model of problem
4. translate
5. check
Checking of feasibility
6. decide
No
Yes = solved!
mathematics
Modeling
reasoning
MATH MODELING
BUILDING THE MODEL:
What is Mathematical Modeling?
Real World
Mathematical World
Observed Data
Mathematical Model
Predicted Behavior
Observed Data of the Future
Compare!
What is mathematical modeling?
Mathematical World
Mathematical techniques
Equations
Computer
Solutions
Problems
Answers
Clinical Practice
Price
Quantity
Demand
Supply
Mathematical Modeling
Types of Models
Mathematical Representation
Processing
Modeling
Interpreting
Physical System
Evaluating
Math Modeling
What is a Math Model?
Mathematical Modeling
What is a Math Model?
MODELS TYPE
Scientific Question
Check
Derive equations
Scientific Result
Mathematical Model
Mathematical Solution
Interpret
Analyze
MATHEMATICAL MODELING
REAL-WORLD PROBLEM
REPORTING RESULTS
Mathematics
Real-World Context
Problem
Formulate
Validate
Report
Compute
Interpret
Modified from Common Core State Standards for Mathematics, page 72.
System
MATHEMATICAL MODELING WITH EXCEL
Mathematical modeling
Mathematical modeling is the art of capturing natural phenomenon of real life in the mathematical equation.
Example: bike speedometer shows 70 km/hr
K. K. TUNG
Topics in Mathematical Modeling
Mathematical Modeling
Mathematical Modeling in Systems Biology
Mathematical Modeling with Multidisciplinary Applications
Problem
Formulate
Validate
Report
Compute
Interpret

So my friendly suggestion is that we think of modeling as a process of making sense about real world phenomena that includes:

Reflecting on “what counts” in the situation at hand;

Building representations thereof;

Operating on the representations, being constantly mindful of the reality and the representations;

Constantly doing theory and reality testing,

Revising until we feel we’re “close enough”, and

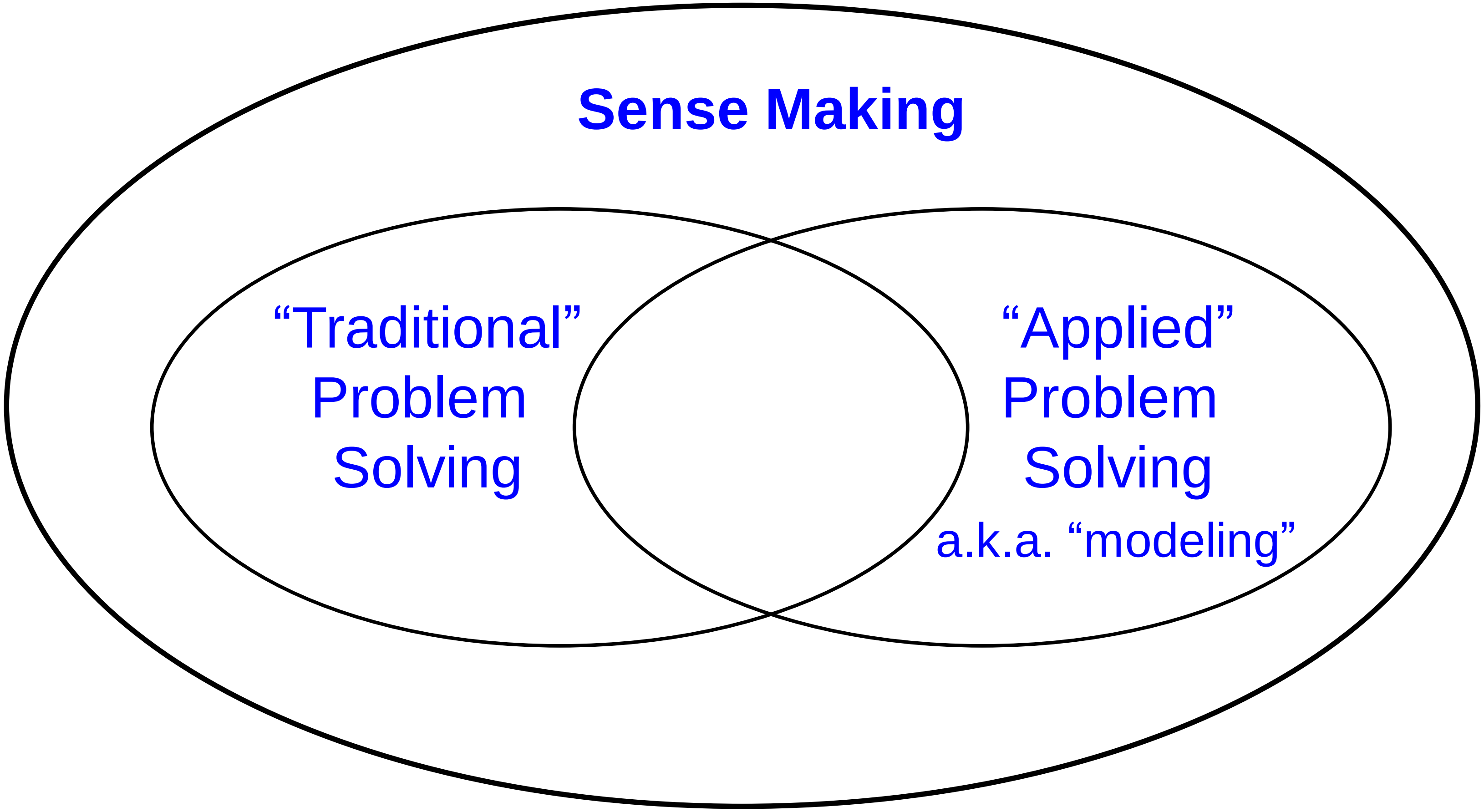
Doing all of this as reflectively as we can.

Guess what?

This is fundamentally a form of sense making, in which the objects we're making sense of happen to come from the real world as well as from the world of mathematics.

That is, here's how I view the entire enterprise:

Whether what you're doing is pure or applied, what matters is thinking things through with the help of the appropriate mathematical tools.



I hope I have been adequately provocative, and I look forward to your reactions.